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Droplet theory in low dimensions: Potts models and percolation

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Abstract. A recently developed droplet theory of low-dimensional Ising systems is generalised to describe the q -state Potts universality class, and thence the percolation problem realised in the $q \rightarrow 1$ limit. An integral equation for the Potts model free energy is derived and used to obtain the generating function for the cluster size probability distribution in the percolation problem, above the percolation threshold. A closed equation for the distribution is obtained and shown to yield scaling behaviour. The form of the distribution is determined analytically in the regimes of small and large cluster sizes yielding, respectively, power law decay and Kunz–Souillard exponential decay. The form of the distribution in the regime of intermediate cluster sizes is determined numerically in space dimension $d = 2$.

1. Introduction

The configurational physics of low-dimensional Ising systems has been extensively studied in two recent papers (Bruce and Wallace 1983, Sim and Bruce 1985, hereafter referred to as I and II, respectively). The essential conclusion of these studies is that, in space dimension $d = 1 + \varepsilon$, the droplet picture of Ising configurations initiated by Fisher (1967) and refined by Kadanoff (1976) is realised in an explicit and analytically controllable way: the configurations consist of droplets of the two phases whose doubly fractal geometry (surface area and connected volume) can be directly linked to the character of critical point singularities.

In this paper we extend this programme of studies to the universality class of q -state Potts models (Potts 1952, Domb 1974) and thence to the problem of bond percolation, the essential physics of which is captured by the $q \rightarrow 1$ limit of the Potts model free energy (Fortuin and Kasteleyn 1972). The structure of the paper is as follows.

In § 2 we indicate the straightforward generalisations of the Ising (two-phase) calculations needed to accommodate the q -phase problem, and derive the key result of the paper: an integral equation satisfied by the field-dependent free energy. We establish the implied critical behaviour of thermodynamic properties and extend the analysis to determine the form of the critical singularity exhibited by the mean droplet size.

In § 3 we turn to the percolation problem. We recall the relationship between the $q \rightarrow 1$ limit of the Potts model free energy and the generating function for the cluster probability distribution describing bond percolation: the results of § 2 then immediately yield an integral equation for the generating function, valid for $d = 1 + \varepsilon$ and $p > p_c$. We establish that the implied distribution function has the anticipated scaling form

and identify the key exponents (β and γ) characterising its low order moments. We proceed to solve analytically for the limiting behaviour of the cluster probability distribution for small and for large clusters, obtaining in the former case the anticipated pure power law fall-off and in the latter case a properly scaling version of the general exponential form suggested by Kunz and Souillard (1978a, b); this result is also established by an independent and physically transparent argument outlined in the appendix. The form of the distribution at intermediate cluster sizes is obtained by solving numerically the renormalisation group differential equation it obeys in $d = 2$. Finally, in § 4, we review the implications and limitations of our results: specifically, we discuss the light which they cast on the percolation phenomenon implicit in the droplet theory of the Ising problem, and on the results of Monte Carlo studies of the cluster probability distribution in two space dimensions (Nakanishi and Stanley 1980).

2. Droplet theory of the Potts model

2.1. The free energy

In this section we determine a droplet-based representation for the free energy of the q -state Potts universality class. The notation and spirit of the arguments will closely follow those of the detailed analysis of the Ising case given in II: accordingly we shall emphasise only the generalisations necessary to accommodate q (rather than two) phases, referring the reader to II for further detail.

The configurational energy required to embed a droplet of one phase (α , say) in a background formed from a different phase (β , say) is written as the sum of two terms

$$\mathcal{H} = T_0^{-1} \mathcal{A} + \frac{hq}{q-1} (\delta_{\beta 1} - \delta_{\alpha 1}) \mathcal{V}. \quad (2.1)$$

The first term represents the energy associated with the surface separating the two phases; it is proportional to the surface area \mathcal{A} of the droplet with a proportionality constant T_0^{-1} which, in keeping with the symmetry of the Potts model, we take to be independent of the specific phases (α and β) involved. This term thus has precisely the same form as in the Ising case, as defined explicitly in II, equation (2.2). The second term represents the energy arising from an ordering field which is chosen to favour one particular phase (labelled 1); it is directly proportional to the volume \mathcal{V} of the droplet, defined explicitly in II, equation (2.1). To understand the structure of the prefactor one must appreciate that equation (2.1) is designed to identify the energy of a single droplet configuration *relative* to that of a droplet-free configuration, so that the field term vanishes when neither α nor β coincides with the favoured phase index. Finally the normalisation of the prefactor is chosen so that the Potts order parameter (defined to be unity at low temperatures) is given by the derivative, with respect to the field h , of the assembly free energy density to be determined below.

The functional integral over all single droplet configurations, weighted by the Boltzmann factor implied by (2.1), yields the single droplet partition function Z_1 . A one-loop calculation yields the result (a trivial generalisation of the Ising case, II equation (2.12))

$$Z_{1,\alpha,\beta} = V_\beta \int dR v^{-1}(R) \psi(R, \xi) \exp\left(\frac{hq}{q-1} (\delta_{\alpha 1} - \delta_{\beta 1}) v(R)\right) \quad (2.2)$$

where V_β is the volume of the embedding space, $v(R) = d^{-1}S_d R^d$ is the volume of a hypersphere of radius R and ξ is the correlation length. The function $\psi(R, \xi)$ is defined in equation (4.4) of I. For the present purposes it is sufficient to recall its general characteristics. Firstly, it has the scaling form

$$\psi(R, \xi) = R^{-1} \tilde{\psi}(R/\xi). \tag{2.3a}$$

Secondly, for droplet scale sizes R small compared to ξ ,

$$\tilde{\psi}(z) \approx \psi_0 \tag{2.3b}$$

where ψ_0 is a universal constant (prescribed in a one-loop calculation by I, equation (4.4d); for a two-loop calculation see Schmittmann (1984)). Finally, for droplet sizes large compared to ξ ,

$$\tilde{\psi}(z) \approx \psi^{(\infty)} z^{(d^2-1)/2} \exp[-(S_d/c_0)z^{d-1}] \tag{2.3c}$$

where the argument of the exponential is identifiable as the interfacial free energy of a spherical droplet of radius R . The parameter c_0 is a dimensionless factor defined to bring into precise coincidence the critical length ξ and the true correlation length characterising the exponential decay of the correlation function (see Bruce (1984) for elaboration of this point). Within the dilute droplet boundary approximation (discussed in the Ising context in I, and in a preliminary study of the Potts problem by Schmittmann (1982)) the single droplet partition function (2.2) is sufficient to prescribe the partition function for an assembly consisting of droplets of all possible phases (q in number) and all possible length scales (bounded between some arbitrary inner length scale L_0 , simulating a lattice cutoff, and the system size L). Following II we find that the result may be expressed as an integro-differential equation for the dimensionless free energy density:

$$\begin{aligned} f(h, L_0, \xi) = & h + \frac{(q-1)}{q} \int_{L_0}^{\infty} dR v^{-1}(R) \psi(R, \xi) \\ & \times \left[[(q-1)Q(h, R, \xi) + 1] \exp\left(-\frac{q}{q-1} h v(R)\right) \right. \\ & \left. + [1 - Q(h, R, \xi)] \exp\left(\frac{q}{q-1} h v(R)\right) + [1 - Q(h, R, \xi)](q-2) \right] \end{aligned} \tag{2.4a}$$

where

$$Q(h, R, \xi) = \frac{\partial}{\partial h} f(h, R, \xi) \tag{2.4b}$$

and we have taken the thermodynamic ($L \rightarrow \infty$) limit.

Equation (2.4a) is the key result of this section and forms the essential basis for the rest of the paper. It is a direct generalisation of the expression for the Ising free energy, given in II, equation (3.9), and is expected to have a similar domain of validity (see II, § 6 and § 3.1 below). Its essential structure is easily appreciated: the first contribution to the integrand can be traced to the dressing of the field-favoured phase, 1, by droplets of the $q-1$ unfavoured phases; the second term originates in the dressing of the unfavoured phases by droplets of phase 1; the final term reflects the dressing of each of the $(q-1)$ unfavoured phases by droplets of any of the *other* $(q-2)$ unfavoured phases.

2.2. The order parameter, susceptibility and mean droplet size

Expressions for the various thermodynamic observables follow immediately from equation (2.4). Specifically, the order parameter (the zero field magnetisation) $Q(h=0, L_0, \xi)$ satisfies

$$Q(0, L_0, \xi) = 1 - q \int_{L_0}^{\infty} dR \psi(R, \xi) Q(0, R, \xi) \quad (2.5a)$$

which has the solution

$$Q(0, L_0, \xi) = \exp\left(-q \int_{L_0}^{\infty} dR \psi(R, \xi)\right) \quad (2.5b)$$

with the asymptotic critical behaviour (cf equations (2.3a, b))

$$Q(0, L_0, \xi) \sim (\xi/L_0)^{-\beta/\nu} \quad (2.6a)$$

with

$$\beta/\nu = q\psi_0. \quad (2.6b)$$

The susceptibility may be obtained in a similar fashion with the asymptotic result

$$\chi(0, L_0, \xi) \sim (\xi/L_0)^{\gamma/\nu} \quad (2.7a)$$

with

$$\gamma/\nu = d - 2q\psi_0. \quad (2.7b)$$

Equations (2.6b) and (2.7b) manifest the strong scaling built into our droplet theory.

One further result will subsequently prove interesting: we identify the exponent θ which characterises the rate of divergence of the mean droplet size, prescribed by the second moment of the droplet size distribution (cf I, equation (6.7))

$$S = \int_{L_0}^{\infty} dR n(R, \xi) V^2(R, L_0, \xi) \quad (2.8)$$

where $V(R, L_0, \xi)$ is the connected (dressed) volume of a droplet of scale size R and $n(R, \xi) dR$ represents the mean number of droplets of scale sizes $R \rightarrow R + dR$ in zero field. These functions are readily determined by straightforward extensions of the arguments leading to the free energy, equation (2.4) (cf II, § 5 and Schmittmann (1982)). The asymptotic behaviour of S is then found to be

$$S \sim (\xi/L_0)^{\theta/\nu} \quad (2.9a)$$

with

$$\theta/\nu = d - 2(q-1)\psi_0 = d - 2 \frac{(q-1)\beta}{q\nu}. \quad (2.9b)$$

This result is independent of which class (phase) of droplets the function $n(R, \xi)$ is chosen to characterise. In the Ising limit ($q=2$), we recover the result $\theta = d\nu - \beta = \gamma + \beta$ obtained in I.

One final remark is appropriate at this juncture. All the results of this section are parametrised in terms of the length ξ . The critical behaviour of this length scale is prescribed in the course of the calculation leading to the form (2.2) for the *single*

droplet partition function (cf I, equations (3.28))

$$\xi \sim (T_c - T)^{-\nu} \quad 1/\nu = \varepsilon + \frac{1}{2}\varepsilon^2 + O(\varepsilon^3). \tag{2.10}$$

These results are thus necessarily *independent* of q , a point to which we shall return in § 4.

3. Droplet theory of percolation

3.1. The cluster probability generating function

We now turn to the problem of percolation. Consider a d -dimensional cubic lattice in which nearest-neighbour bonds are present with a probability p . Denote by $p(s)$ the probability that a site belongs to an s -site cluster (a cluster consisting of s sites linked by bonds). We associate with the probability distribution $p(s)$ a generating function $g(h)$ defined such that

$$g(h) = 1 - \sum_{s \geq 1} p(s) e^{-hs} \tag{3.1}$$

or

$$p(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dh e^{ihs} (1 - g(ih)). \tag{3.2}$$

With the understanding that the probability distribution $p(s)$ incorporates only finite clusters we see immediately from equation (3.1) that the probability P that a site belongs to the infinite cluster has the representation

$$P = 1 - \sum_{s \geq 1} p(s) = g(h = 0). \tag{3.3}$$

This probability defines the order parameter for the percolation problem (see Stauffer (1979) and Essam (1980) for recent reviews), an identification which suggests a clear analogy between the role of the generating function in the percolation problem and the role of the free energy in a thermally driven phase transition. This analogy finds a precise expression in the mapping devised by Fortuin and Kasteleyn (1972) who showed that the generating function is related to the free energy density of a Potts model (on an identical lattice) via

$$g(h) = \lim_{q \rightarrow 1} (q - 1)^{-1} \frac{d}{dh} f((q - 1)h) \tag{3.3a}$$

with the auxiliary condition that the bond occupation probability p in the percolation problem and the dimensionless bond-breaking energy K in the Potts problem are related by

$$p = 1 - e^{-K}. \tag{3.3b}$$

The different status of equations (3.3a) and (3.3b) merits comment before we proceed further. Equation (3.3b) expresses a link between two specific lattice models; as such it has no immediate implications for universal quantities (the critical constants K_c and p_c are certainly lattice-dependent) and is of no use to us here. Equation (3.3a), on the other hand, although *derived* within the framework of a lattice model, clearly transcends this limitation: it expresses a link between two universality classes, and

leads to the expectation that the universal behaviour of the generating function $g(h)$ (notably the exponents characterising the singular behaviour of its derivatives and appropriate combinations of associated amplitudes) will be correctly captured by equation (3.3a), no matter what particular realisation of the Potts model is utilised in the calculation of the free energy density f .

Proceeding on this assumption we substitute into equation (3.3a) our droplet-based representation (2.4) for the Potts free energy density. Taking the prescribed $q \rightarrow 1$ limit we find

$$g(h, L_0, \xi) = 1 + \int_{L_0}^{\infty} dR \psi(R, \xi) [(1 - g(h, R, \xi)) e^{hv(R)} - e^{-hv(R)} - v^{-1}(R)g'(h, R, \xi)(e^{hv(R)} - 1)] \tag{3.4a}$$

where

$$g'(h, R, \xi) = (\partial/\partial h)g(h, R, \xi). \tag{3.4b}$$

These equations define an implicit representation of the generating function for the percolation probability distribution $p(s, L_0, \xi)$. A number of remarks are in order. Firstly, the dependence upon the minimum length scale L_0 indicated in equation (3.4a) is a reminder that our representation can be trusted to capture correctly only the truly universal features of the number distribution, whose signature will be their independence of this arbitrary inner length. Secondly, our inclusion of the length ξ amongst the arguments of the generating function anticipates the fact that the results we shall obtain will be parametrised by this length, whose behaviour near the percolation threshold p_c follows from (2.10) and (3.3b) as

$$\xi \sim (p - p_c)^{-\nu} \tag{3.5}$$

with ν as prescribed in equation (2.10) (cf, however, the caveat registered in § 4). Thirdly, we observe that, as noted in II, the representation (2.4) is trustworthy *below* the critical point for fields *small* on the scale of $\xi^{-d}(\xi/L_0)^{\beta/\nu}$. The former limitation implies that (3.4) is trustworthy only *above* the percolation threshold; the latter limitation does not invalidate the use of $g(h)$ as a generating function since, in this capacity, $g(h)$ is required only for infinitesimal real h .

Finally it may forestall some confusion to recall that, within our framework, the field h has dimensions of an inverse volume, since it was chosen (equation (2.1)) to be thermodynamically conjugate to the droplet *volume* rather than to the *number* of sites within the droplet. As a corollary the function g defined in equation (3.4) will act as a generator for ‘ s -sized’ clusters where s itself has the dimensions of volume.

3.2. The moments of the cluster probability distribution

The moments of the cluster number distribution are prescribed by the derivatives of the generating function g with respect to the ‘field’ h . Specifically

$$\langle s^n \rangle = \int_0^{\infty} ds s^n p(s, L_0, \xi) = (-1)^{n+1} [P_n(L_0, \xi) - \delta_{n,0}] \tag{3.6a}$$

where we define

$$P_n(L_0, \xi) = \frac{\partial^n}{\partial h^n} g(h, L_0, \xi) |_{h=0}. \tag{3.6b}$$

An equation for the functions P_n is easily derived from the integral equation (3.4), as discussed for the corresponding Ising problem in II, § 4.1. Neglecting terms which modify amplitudes by multiplicative factors of the order of $1 + \psi_0$ we find (for $n > 0$)

$$\begin{aligned}
 P_n(L_0, \xi) = & \exp\left(- (n+1) \int_{L_0}^{\infty} dR \psi(R, \xi)\right) \\
 & \times \int_{L_0}^{\infty} dR \left[\psi(R, \xi) v^n(R) [1 + (-1)^{n+1} - P_0(R, \xi)] \right. \\
 & \left. \times \exp\left((n+1) \int_R^{\infty} dR' \psi(R', \xi)\right) \right] \quad (3.7a)
 \end{aligned}$$

with

$$P_0(L_0, \xi) = \exp\left(- \int_{L_0}^{\infty} dR \psi(R, \xi)\right) \quad (3.7b)$$

which, with the aid of equations (3.3) and (3.6), we identify with the percolation order parameter P . The implied behaviour near the percolation threshold follows immediately

$$P_0(L_0, \xi) \sim (L_0/\xi)^{\beta/\nu} \quad (3.8a)$$

with

$$\beta/\nu = \psi_0. \quad (3.8b)$$

The behaviour of the *mean* cluster size follows as the second moment of the number distribution which one may obtain readily from equation (3.7):

$$\langle s \rangle = \int_0^{\infty} ds s p(s, L_0, \xi) = P_1(L_0, \xi) \sim (\xi/L_0)^{\gamma/\nu} \quad (3.9a)$$

where

$$\gamma/\nu = d - 2\psi_0 = d - 2(\beta/\nu). \quad (3.9b)$$

Both (3.8b) and (3.9b) have interesting implications for the Ising problem, as we shall return to discuss in § 4. The behaviour of the general n th-order moment follows in a similar way:

$$P_n(L_0, \xi) \sim (\xi/L_0)^{-(n+1)\psi_0} \xi^{nd}. \quad (3.10)$$

The fact that the exponent of the n th order terms varies linearly with n guarantees scaling for the generating function. Specifically, combining equations (3.6) and (3.10) we find

$$g(h, L_0, \xi) = \sum_{n=0}^{\infty} P_n(L_0, \xi) h^n / n! = \xi^{-d+\beta\delta/\nu} \tilde{g}(h\xi^{\beta\delta/\nu}) \quad (3.11a)$$

with

$$\delta = d/\psi_0 - 1 = \gamma/\beta + 1. \quad (3.11b)$$

3.3. The structure of the cluster probability distribution

To investigate the form of the cluster probability distribution we substitute into the

defining relation

$$p(s, L_0, \xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dh e^{ihs} (1 - g(ih, L_0, \xi)) \tag{3.12}$$

the integral representation for the generating function, equation (3.4). Performing the integral on h we find

$$p(s, v_0, \xi) = p_0(s, \xi) \theta(s - v_0) + sd^{-1} \int_{v_0}^{\infty} dv v^{-2} \phi(v, \xi) [p(s + v, v, \xi) - p(s, v, \xi)]. \tag{3.13a}$$

We have changed the variable of integration from R to $v(R)$ and, correspondingly, have chosen to reparametrise the short-distance cutoff implicit in the probability distribution by a minimum volume scale $v_0 = v(L_0)$ rather than the minimum length scale L_0 itself. We have also introduced the abbreviations

$$p_0(s, \xi) \equiv (ds)^{-1} \phi(s, \xi) \tag{3.13b}$$

$$\phi(v, \xi) = \tilde{\psi}((vd/S_d)^{1/d}/\xi).$$

Equation (3.13a) is the central result of this paper. It can be solved analytically in the regions of sufficiently small, and sufficiently large, s values, as we now proceed to show.

We begin by recasting the equation in the differential form

$$\frac{\partial p(s, v, \xi)}{\partial v} = -p_0(s, \xi) \delta(s - v) - d^{-1} v^{-2} s \phi(v, \xi) [p(s + v, v, \xi) - p(s, v, \xi)]. \tag{3.14}$$

Now we observe that in the ‘scaling’ regime characterised by s values large compared to the ultraviolet cutoff v the difference on the RHS of equation (3.14) may be replaced by a derivative with respect to the cluster size, yielding the partial differential equation

$$\frac{\partial p(s, v, \xi)}{\partial v} = -d^{-1} v^{-1} s \phi(v, \xi) \frac{\partial p(s, v, \xi)}{\partial s} \quad s \gg v. \tag{3.15}$$

The solutions to this equation are of the form

$$p(s, v, \xi) = p_\xi(s/P_0(v, \xi)) \tag{3.16}$$

where $P_0(v, \xi)$ obeys the differential equation

$$\frac{\partial P_0(v, \xi)}{\partial v} = (dv)^{-1} \phi(v, \xi) P_0(v, \xi) \tag{3.17}$$

which is immediately recognisable as the equation satisfied by the order parameter (equation 3.7b), with the ultraviolet cutoff reparametrised as indicated above.

Thus far in the argument we have invoked no approximation other than that imposed by the requirement $s \gg v$ needed to justify the replacement of (3.14) by (3.15). Now let us focus specifically on the region of large s . By ‘large’ we mean, in the first instance, s values for which $s \gg v(\xi) \sim \xi^d$, although this condition will subsequently need to be refined. Given this additional constraint one sees, firstly, that equation (3.16) may be invoked even for v values as large as $\alpha v(\xi)$, with α some suitable small constant. Thus, explicitly,

$$p(s, \alpha v(\xi), \xi) = p_\xi(s/P_0(\alpha v(\xi), \xi)). \tag{3.18a}$$

Moreover, we see from equations (3.13a) and (3.7b) that, to within exponentially small corrections,

$$p(s, \alpha v(\xi), \xi) = p_0(s, \xi) \tag{3.18b}$$

$$P_0(\alpha v(\xi), \xi) = 1. \tag{3.18c}$$

Matching equations (3.18a, b, c) we conclude that

$$p_\xi(s) = p_0(s, \xi) \tag{3.19}$$

whence, substituting back into equation (3.16), and setting $v = v_0$,

$$p(s, v_0, \xi) = p_0(s/P_0(v_0, \xi), \xi) \quad s \gg v(\xi). \tag{3.20}$$

In the appendix we show that this result may be derived by rather simple independent arguments which reveal its physical significance. Here we make its form more explicit by recalling the large argument behaviour of the function $\tilde{\psi}$ (equation (2.3c); cf also equation (3.13b)), obtaining finally

$$p(s, v_0, \xi) = \psi^{(\infty)}(dv(\xi))^{-1} \tilde{s}^{1-\theta'} \exp\left(-\frac{S_d}{c_0} \tilde{s}^{(d-1)/d}\right) \tag{3.21a}$$

where

$$\theta' = (1 + 4d - d^2)/2d \tag{3.21b}$$

and \tilde{s} is a scaled cluster size variable

$$\tilde{s} = s[P_0(v_0, \xi)v(\xi)]^{-1}. \tag{3.21c}$$

Now let us consider the regime of 'small' s values, i.e. in the first instance, $s \ll v(\xi)$. In this case the differential equation (3.15) and the functional form (3.16) which it implies for the cluster probability distribution are not trustworthy in the region $v \sim v(\xi)$, since the 'differential' approximation made in replacing (3.14) by (3.15) clearly breaks down at $v = s$ ($\ll v(\xi)$): accordingly we must search for some other regime of v values, which does lie within the domain of validity of the differential equation (3.15), and for which we may prescribe the form of the probability function. This aim is realised with the choice $v = \alpha s$, with α some suitably small constant. Clearly such a choice guarantees the validity of the differential approximation, and thus of the form (3.16), for this v value:

$$p(s, \alpha s, \xi) = p_\xi(s/P_0(\alpha s, \xi)). \tag{3.22a}$$

Moreover, and less trivially, we assert that for such a choice the probability distribution has the form

$$p(s, \alpha s, \xi) = A(\alpha)/s \tag{3.22b}$$

where $A(\alpha)$ is some α -dependent amplitude which we need not prescribe. To justify this assertion we consider an alternative approach to the solution of equation (3.13a): one may, in fact, proceed to develop a hierarchy of approximate solutions to this equation by iteration in powers of the concentration function ϕ , taking as the first-order solution the function

$$\begin{aligned} p_1(s, v_0, \xi) &\equiv p_0(s, \xi)\theta(s - v_0) \\ &\approx (ds)^{-1}\psi_0\theta(s - v_0) \quad s \ll v(\xi). \end{aligned} \tag{3.23}$$

In the small s region one discovers that the iteration procedure generates a perturbation expansion in powers of $\psi_0 \ln(s/v_0)$. This expansion is actually recognisable as the ψ_0 expansion of a power law in s ; however, for the present purposes it is sufficient to note that, replacing the ultraviolet cutoff v_0 by αs quells the logarithmically divergent corrections to the leading (analytic) behaviour displayed by the first-order solution (3.23), so that these corrections merely renormalise the amplitude of the s^{-1} behaviour. Equation (3.22*b*) is thus vindicated. Equations (3.22*a*) and (3.22*b*) may now be combined to yield

$$p_\xi(s/P_0(\alpha s, \xi)) \approx A(\alpha)/s. \tag{3.24a}$$

Now, with the present parametrisation, the order parameter has the behaviour (cf equation (3.8))

$$P_0(v, \xi) = B(v/\xi^d)^{\psi_0/d} \tag{3.24b}$$

where B is a constant. Utilising this result in conjunction with equation (3.24*a*) we find

$$p_\xi(x) \approx A(\alpha) [xB(\alpha/\xi^d)^{\psi_0/d}]^{-d/(d-\psi_0)}. \tag{3.24c}$$

Combining this result with equation (3.16) we obtain

$$p(s, v_0, \xi) \approx A(\alpha)(\alpha/v_0)^{-\psi_0/(d-\psi_0)} s^{-d/(d-\psi_0)}. \tag{3.25}$$

The self-consistency of the argument requires that the result be independent of the specific choice of the matching parameter α so that, finally,

$$p(s, v_0, \xi) \approx As^{-1}(s/v_0)^{-\psi_0/(d-\psi_0)} \quad s \ll v(\xi) \tag{3.26a}$$

where $A \equiv A(1)$ is an amplitude which is independent of the ultraviolet cutoff v_0 , but which this argument does not otherwise prescribe. In fact the perturbative (iterative) approach to equation (3.13*a*), discussed above, shows that

$$A = \psi_0/(d - \psi_0) + O(\psi_0^3). \tag{3.26b}$$

Inspection of the two limiting cases (3.20) and (3.26) reveals that they are both consistent with the general scaling form

$$p(s, v_0, \xi) \approx (s/B_0)^{1-\tau} \tilde{p}(s/(B_0 s_\xi)) \tag{3.27a}$$

where

$$\tau = 2 + \psi_0/(d - \psi_0) = 2 + 1/\delta \tag{3.27b}$$

and

$$s_\xi = v(\xi)\xi^{-\psi_0} = S_d d^{-1} \xi^{d-\beta/\nu} \tag{3.27c}$$

while B_0 is a non-universal parameter defining the critical behaviour of the order parameter:

$$P_0(v_0, \xi) = B_0 \xi^{-\beta/\nu}. \tag{3.27d}$$

Written in this way the scaling function \tilde{p} is universal, given a specific convention for the definition of the correlation length.

Two remarks are now in order. Firstly we observe that one may establish the scaling form (3.27) more generally, and at the same time refine it somewhat, by substituting the scaling form of the function g (equation (3.11)) into the defining relation for the probability distribution, equation (3.12). The scaling form (3.27) is then found to hold

in the critical region $\xi^d \gg v_0$ (i.e. $\xi \gg L_0$), where the moments scale according to equation (3.10), modulo an additional non-scaling and analytic contribution which contributes significantly only to the zeroth moment of the probability distribution function.

Secondly, equation (3.27) makes it plain that the dividing line between the ‘small s ’ and ‘large s ’ regions examined explicitly above, is actually set by the ‘typical cluster size’ (equation (3.27c)) rather than by $v(\xi)$. This state of affairs is reflected in the form of the perturbation expansion leading to equation (3.26b), through the appearance of terms which, though smaller than the leading contributions by powers of $s/v(\xi)$, carry additional factors of $\ln(s/v_0)$.

To complement the small s and large s solutions to equation (3.13a) determined analytically above we have obtained solutions for intermediate values of s by numerical integration of equation (3.14), from the regime $v \gg v(\xi)$ where the first-order perturbative solution $p_1(s, v, \xi)$ is satisfactory down into the scaling regime $v = v_0 \ll v(\xi)$. In performing the calculations we have set $d = 2$ and have parametrised the function $\tilde{\psi}$ such that the amplitude $\psi^{(\infty)}$ is given by II, equation (4.14). We have chosen the parameters $\psi_0 = \beta/\nu$ and ν to reproduce the $d = 2$ values of the percolation indices as given by series studies (reviewed by Stauffer (1979)); we have set $c_0 = 2$, in accordance with a conjecture developed elsewhere (Bruce 1984) which presupposes the convention that ξ is that length which characterises the exponential falloff of the pair-connectedness function; finally we have (cf II, equation (4.14)) set the parameter $T_c = \varepsilon$ (at one loop) = 1. We defer further comment on these assignments to the following section.

The results of these calculations are presented in figure 1 which shows the logarithm of the scaling function \tilde{p} , defined by equations (3.27), plotted as a function of the square root of the scaling variable (cf equations (3.27a) and (3.21c))

$$\tilde{s} = s [P_0(v_0, \xi) \pi \xi^2]^{-1}. \tag{3.28}$$

Clearly the function evolves in a relatively structureless fashion between the limiting small \tilde{s} value, effectively prescribed by the perturbative result (3.26b), and the large \tilde{s} behaviour

$$-\ln \tilde{p}(\tilde{s}) \approx \pi \tilde{s}^{1/2} - 0.804 \ln \tilde{s} + \text{constant} \tag{3.29}$$

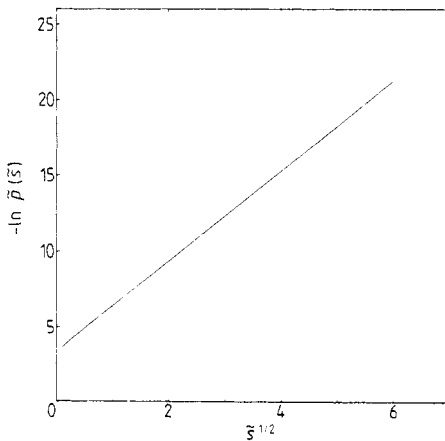


Figure 1. The negative of the logarithm of the $d = 2$ scaling function \tilde{p} for the cluster probability (equation 3.27a) plotted against the square root of the scaling variable \tilde{s} (equation 3.28).

following from equation (3.21). We remark, however, that the curvature of the function changes sign at $\tilde{s} \approx 2$. For higher \tilde{s} values the curvature is positive in accordance with the asymptotic form (3.29). For lower \tilde{s} values the curvature is negative. Indeed we believe that, as a function of \tilde{s} , \tilde{p} approaches its $\tilde{s} = 0$ value non-analytically (with infinite slope). Although we have not investigated this feature in detail it seems likely that it should be attributed to the fact that, for fixed droplet scale size $R \ll \xi$ the droplet population function $\tilde{\psi}(R/\xi)$ approaches ψ_0 non-analytically as a function of (increasing) ξ , though analytically as a function of (decreasing) $t \sim \xi^{-1/\nu}$ (cf I, equation (4.4c)).

4. Discussion

We begin our discussion by examining the lessons to be learned from the predictions assembled in this paper for the forms of various critical indices.

The simplest and yet the most manifestly unsatisfactory prediction is that for the correlation length index ν , equation (2.9). The q independence of this prediction is at variance with established results: the $q = 1$, percolation, value in $d = 2$ is $\nu \approx 1.35$ (Stauffer 1979) while for $q = 2$ we have the exact Onsager result $\nu = 1$. This deficiency is an immediate corollary of the fact that this index is derived from the behaviour of an artificially *isolated* interface: the theory takes no account of the effects of other droplets upon interfacial fluctuations, effects which may include the decoration of an interface separating two phases by droplets of a third phase (Selke and Pesch 1982). We expect that such effects will give rise to corrections to the index ν that are of the order of $(q - 2)\psi_0^{1/2}$; thus the interface-based calculations, which have been extended to 4-loop order (Forster and Gabriunas 1981) may well be most satisfactory in the Ising ($q \rightarrow 2$) limit. At this point we should register a caveat concerning equations (2.10) and (3.5). These results suggests that, in the $d \rightarrow 1$ limit, the percolation index ν diverges as $1/(d - 1)$ whereas, in fact, in $d = 1$ the value of ν (for the percolation of nearest-neighbour connected clusters) is $\nu = 1$ (Klein *et al* 1978). This apparent discrepancy is resolved by the recognition that equation (3.5) presupposes that the deviation $p - p_c$ in the percolation problem has a Taylor series expansion in the deviation $K - K_c$ for the corresponding Potts model. Equation (3.3b) shows that this assumption is valid for $d \neq 1$. For $d = 1$, however, where $K_c^{-1} \equiv 0$, we have $p_c = 1$ and $p - p_c \sim e^{-K} \sim \xi^{-1}$, implying $\nu = 1$ in accord with the known result.

Let us turn now to the predictions (2.6b) and (3.8b) for the order parameter exponent β . These results illuminate an interesting feature of the Ising droplet model, remarked on in I (§ 8). In the Ising problem one finds that as the critical point is approached from below the volume of that portion of the originally homogeneously ordered phase left after decoration by the droplet hierarchy vanishes as $\xi^{-(\beta/\nu)_I}$ where

$$(\beta/\nu)_I' = \frac{1}{2}(\beta/\nu)_I. \quad (4.1a)$$

(When necessary we will use subscripts I and P to distinguish between Ising and percolation exponents.) The recognition (cf equations (2.6b) and (3.8b)) that

$$(\beta/\nu)_I = 2(\beta/\nu)_P \quad (4.1b)$$

thus reveals that

$$(\beta/\nu)_I' = (\beta/\nu)_P. \quad (4.1c)$$

The implication is clear: implicit in the droplet representation of the Ising problem is

a pure ('random') percolation problem in which the role of the infinite cluster is played by the residual undecorated volume of the ground-state phase.

This implication is also reflected in our result for the Ising droplet size exponent (equation (2.9*b*))

$$(\theta/\nu)_I = d - (\beta/\nu)_I \quad (4.2a)$$

which, when combined with the standard scaling result for the cluster size exponent in the percolation problem (equation (3.9*b*))

$$(\gamma/\nu)_P = d - 2(\beta/\nu)_P \quad (4.2b)$$

yields, in conjunction with equation (4.1*b*),

$$(\theta/\nu)_I = (\gamma/\nu)_P. \quad (4.2c)$$

Thus the mean droplet size (more precisely, the second moment of the droplet size distribution) in the Ising problem diverges with the same power of the thermal correlation length as does the mean cluster size in the percolation problem as a function of the connectedness length.

Regrettably, the link between Ising and percolation behaviour is not as simple as these results would suggest. Certainly, the Ising critical point *is* a percolation point, in $d = 2$ (Coniglio *et al* 1977), and the predictions (4.1*a*) and (4.2*a*) are not inconsistent with the existing numerically established results (cf Sykes and Gaunt (1976), Jan *et al* (1982) and further discussion in I). However the equations proffering explicit links between percolation and Ising indices (equations (4.1*b*) and (4.2*c*)) are irreconcilable with series-based results for percolation exponents (Stauffer 1979). It is possible to envisage a refined theory in which droplet decoration of interfaces gives a q -dependent renormalisation of both ν and ψ_0 , while leaving the structure of the theory of the many-droplet assembly essentially unaltered: the identifications (4.1*a*) and (4.2*a*) would then survive, while the erroneously simple relationships between exponents associated with systems of different q ((4.1*b*) and (4.2*c*)) would be modified. It is also possible, however, that the noted inconsistencies are an indication that the neglect of partial droplet overlap (the $O(\psi_0)$ approximation) inherent in our theory of the many-droplet assembly represents an unexpectedly severe limitation already in $d = 2$.

In any event let us now turn to consider the key results of this paper, namely those concerning the cluster size probability distribution. We have seen, first of all, that our theory confirms the anticipated scaling form of the cluster distribution: specifically equations (3.27*a*)–(3.27*d*) are in complete accord with the results of purely phenomenological arguments summarised, for example, by Stauffer (1979). One appreciates that this vindication of phenomenological scaling arguments is non-trivial when one recalls that the corresponding phenomenological arguments applied to *droplets* in the Ising problem (Binder 1976) were shown in I to be inconsistent with the form derived from our explicit droplet theory (cf I, § 6).

Secondly, and more significantly, our theory generates a closed equation (3.13*a*) for the cluster probability function. For clusters small compared to the typical cluster size s_ξ (equation (3.27*c*)) the distribution exhibits pure power law decay (equation (3.26)). For clusters large compared to s_ξ the distribution exhibits an exponential fall-off (equation (3.21)) which bears out a prediction first made by Kunz and Souillard (1978*a, b*), refining it in a number of important instances. Specifically, the $s^{(d-1)/d}$

dependence of the argument of the exponential is seen to hold, in the present theory, arbitrarily close to the percolation threshold, a region in which the Kunz-Souillard theorem was not formally proven. This result bears out a conjecture by Klein and Stauffer (1980) and goes some way to allaying the doubts on this issue expressed by Nakanishi and Stanley (1980) on the basis of Monte Carlo studies to which we shall shortly return. As a corollary we see that the Kunz-Souillard form appears with parameters whose ξ dependence ensures consistency with scaling (cf also a renormalisation group argument valid near $d = 6$, cited by Harris and Lubensky (1981)). Indeed the form of the argument of the exponential is readily intelligible in terms of the underlying thermal problem if we note that

$$\tilde{s}^{(d-1)/d} \sim \xi^{-(d-1)} (s\xi^{\beta/\nu})^{(d-1)/d}. \quad (4.3)$$

The first factor is identifiable as a measure of the surface tension; the factor $s\xi^{\beta/\nu}$ is a measure of the volume of space confined within the boundaries of a droplet of connected volume s and fractal dimension (Mandelbrot 1982, Kapitulnik *et al* 1983)

$$d_v = d - \beta/\nu. \quad (4.4)$$

The factor $(s\xi^{\beta/\nu})^{(d-1)/d}$ thus measures the external surface area of such a droplet, and the argument of the exponential is simply the classical free energy of an essentially spherical droplet containing the prescribed connected volume. This view of the significance of equations (3.20) and (3.21) is developed more explicitly in the appendix.

Our calculation also captures the power law prefactor structure associated with the asymptotic (large s) form. This result (specifically, the prediction for the exponent θ' , equation (3.21)) is in accord with a calculation by Lubensky and McKane (1981). Their study, like our own, exploits the Potts representation of the percolation problem, and traces the asymptotic form of the distribution to the effects of large droplets. Their calculation, however, does not incorporate the critical (droplet-nesting) effects which produce a properly scaling form, and thus yields, in effect, simply the first-order solution (3.23a) to our fundamental equation (3.13a). On the other hand, the price (the dependence upon dimensional regularisation in the calculation of $\psi(R, \xi)$: cf I) we have paid to control critical point effects shows up in the failure of our theory to detect the anomaly in the exponent θ' (i.e. the failure of equation (3.21b)) in $d = 3$, observed by Lubensky and McKane whose calculation is performed without recourse to a small $d - 1$ expansion. Indeed it is both gratifying and remarkable that the dimensionally regularised calculation proves correct (in this respect) for $d < 3$.

Finally, let us return to discuss the Monte Carlo studies of the cluster probability distribution (Nakanishi and Stanley 1980) alluded to above. Perhaps the most distinctive (and potentially troublesome) feature displayed by these calculations is their apparent inconsistency with the Kunz-Souillard form. Specifically, close to the percolation threshold a plot of (minus) the logarithm of the cluster probability against the square root of the cluster number shows significant deviations from the linear behaviour expected for 'large' cluster numbers, in $d = 2$ (and confirmed by the present calculations). A number of comments are in order here. Firstly, the existence of the prefactor structure in equation (3.21a) shows that such a plot will, if extrapolated back to $\tilde{s} = 0$, have a finite intercept with the $\tilde{s} = 0$ axis, and will have a positive curvature (decreasing as $1/\sqrt{s}$). The former feature is indicated in the $d = 2$ Monte Carlo data (Nakanishi and Stanley 1980, figure 13a), but the curvature apparent in the data would appear

to be *negative*. However, given the observed limiting slope of the data and utilising equation (3.29) one may estimate the ξ values appropriate for the p values studied (the amplitude connecting ξ and $p_c - p$ has not, to our knowledge, been determined) and thence estimate the range of values of the scaling variable \tilde{s} (equation 3.28) actually studied in the Monte Carlo experiment: one then finds that the troublesome data are actually associated with \tilde{s} values *smaller* than that ($\tilde{s} \approx 2$) at which the curvature in $-\ln \tilde{p}(\tilde{s})$ (as given by the present theory) changes sign. It thus seems quite likely that the Monte Carlo results are, in fact, consistent with the Kunz-Souillard form. Naturally it would be desirable to make a more complete comparison between the Monte Carlo results, and those presented here. However, in view of the relatively structureless character of figure 1, a meaningful comparison of this type must await the determination of the amplitude connecting ξ and $p_c - p$: one could then determine the connection between the observed cluster number s and the scaling variable \tilde{s} (equation (3.28)) and carry through a comparison with no adjustable scale parameters.

With this possibility in mind we conclude by offering our assessment of the trustworthiness of the results summarised in figure 1. The behaviour of the computed scaling function for *small* cluster sizes rests heavily on the dilute-droplet approximation at the heart of our theory; although this dependence is tempered through our parametrisation of ψ_0 by the 'correct' (series) value of the ratio $(\beta/\nu)_p$, the reliability of our predictions in this region remains open to question. By contrast, the validity of the predicted form of the scaling function for *large* clusters very probably transcends that of the dilute-droplet approximation. Specifically, the structure of the argument of the exponential in $\tilde{p}(\tilde{s})$, represented in the first term in equation (3.29), is supported by rather general configurational arguments (cf the appendix and, as regards the value of c_0 , the study by Bruce (1984)). The \tilde{s} -dependent prefactor structure in $\tilde{p}(\tilde{s})$, represented in the logarithmic term in equation (3.29), is also likely to be trustworthy in $d = 2$ where, we believe, our calculation correctly handles the droplet zero modes which control this factor. Finally we observe that the \tilde{s} -independent prefactor in $\tilde{p}(\tilde{s})$, represented in the 'constant' in equation (3.29), is controlled essentially by the parameter $\psi^{(\infty)}$ (equation (2.3c)). This parameter is, we surmise, possibly universal in a strong sense (independent of q) and, as such, might actually be obtained from the results of the field-expansion studies of coexistence curve behaviour in the $d = 2$ Ising model, discussed in II, § 4.1. However, in parametrising the function $\tilde{\psi}$ with the true percolation indices (cf § 3.3) this strong universality of $\psi^{(\infty)}$ has been compromised: the implied value of $\psi^{(\infty)}$ is a factor of two smaller than that suggested by the Ising data examined in II.

Appendix. The large s behaviour of the cluster probability distribution

In this appendix we present an alternative derivation of the large s behaviour of the cluster probability function which clarifies its physical origins. Consider a q -state Potts model on a hypercubic lattice of V sites and unit spacing; we take the system to be below its ordering temperature, with an infinitesimal field favouring phase 1. Denote by M the number of spins which do not occupy the state 1:

$$M \equiv \sum_i (1 - \delta_{\sigma_i, 1}) \quad (\text{A1})$$

where σ_i labels the state of the spin at site i . The probability density $p_q(M)$ for the

number M may be written as

$$\begin{aligned}
 p_q(M) &= \left\langle \delta \left(M - \sum_i (1 - \delta_{\sigma_i,1}) \right) \right\rangle_0 \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dh e^{ihM} \left\langle \exp \left(ih \sum_i (\delta_{\sigma_i,1} - 1) \right) \right\rangle_0
 \end{aligned}
 \tag{A2}$$

where the subscript denotes averaging in the ensemble of zero (infinitesimal) field. Now the Potts model free energy density may be written in the form

$$f(h) = \frac{1}{V} \ln \left\langle \exp \left[\frac{qh}{q-1} \sum_i \left(\delta_{\sigma_i,1} - \frac{1}{q} \right) \right] \right\rangle_0 + f(0)
 \tag{A3}$$

where the argument of the exponential represents the configurational energy associated with the external field. Combining equations (A2) and (A3) with equations (3.2) and (3.3a) one readily finds that

$$p(s) = sV^{-1} \left. \frac{d}{dq} p_q(s) \right|_{q=1}.
 \tag{A4}$$

This result is quite general. It implies, in particular, that the large s behaviour of the cluster probability function $p(s)$ is controlled by the probability of Potts model configurations containing that same large number, s , of spins occupying states other than that which is thermodynamically favoured. It is clear on physical grounds that such configurations will originate in large *effectively spherical* droplets of one of the $q-1$ unfavoured phases. Consider, then, a droplet of phase 2, say, which extends over M_0 sites. Clearly, such a droplet will typically not be fully ordered: for a droplet which is large enough (on the scale of the correlation length) we may expect that only a fraction $(q-1)Qq^{-1} + q^{-1}$ of the M_0 spins will actually be in state 2, where we anticipate that the order parameter for the droplet will coincide with the bulk order parameter:

$$Q = q(q-1)^{-1} \left\langle \delta_{\sigma,2} - \frac{1}{q} \right\rangle_{\text{droplet}} = q(q-1)^{-1} \left\langle \delta_{\sigma,1} - \frac{1}{q} \right\rangle_{\text{bulk}}.
 \tag{A5}$$

The fraction of the spins within the droplet which will be in state 1 is $(q-1)^{-1}[1 - (q-1)Qq^{-1} - q^{-1}]$. Thus the *actual* number M of spins within the droplet that will not be in phase 1 is

$$M = M_0[1 - q^{-1}(1 - Q)].
 \tag{A6}$$

It follows that the probability $p_q(M) dM$ of finding the number of spins in states other than 1 to lie between M and $M + dM$ is equal to the probability of finding a droplet of radius between R and $R + dR$ where

$$M_0 = S_d d^{-1} R^d
 \tag{A7}$$

while M_0 and M are related by equation (A6). Since such large droplets are clearly dilute the required probability may be obtained immediately from equation (2.2):

$$p_q(M) dM \approx (q-1) V v^{-1}(R) \psi(R, \xi) dR.
 \tag{A8}$$

Substituting this result into equation (A4) and utilising equations (A6), (A7) and (2.3a)

we find

$$p(s) \approx (ds/P)^{-1} \tilde{\psi}((ds/S_d P)^{1/d} / \xi) \quad (\text{A9})$$

where we have identified the percolation order parameter P with the $q \rightarrow 1$ limit of the Potts order parameter Q . Equation (A9) recovers precisely the asymptotic form derived in the text (cf equations (3.20) and (3.13b)).

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